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THE HOLONOMY GROUPS AND THE REFINEMENTS OF A PRINCIPAL STEENROD BUNDLE

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Let (ξ, H) be a structure consisting of a principal Steenrod bundles $\xi = (E, B, p, G; \mathscr{A})$ and a closed subgroup H of G. We shall assume that all elements are C^∞ differentiable.

It is well known [2, p. 57] that ξ and H define the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{p_1} & E/H \\ {\scriptstyle p}\downarrow & \swarrow {\scriptstyle p_0} & \\ B & & \end{array}$$

and the Steenrod bundles $\xi_0 = (E/H, B, p_0, G/H, G/N; \mathscr{A}_0)$, $\xi_1 = (E, E/H, p_1, H; \mathscr{A}_1)$. Here N is the largest invariant subgroup of G included in H, p_1 and p_0 are the cannonical maps, and $\mathscr{A}_0$, $\mathscr{A}_1$ are trivialization atlases of ξ_0, ξ_1 respectively, cannonically defined by $\mathscr{A}$, [4]. ξ_1 is a principal Steenrod bundle.

The structure $(\xi; \xi_0, \xi_1)$ will be called the refinement of ξ defined by H, and we shall utilize the following notation:

$\Delta^V, \Delta_1^V, \Delta_0^V$ are the vertical differential systems of ξ, ξ_1, ξ_0 respectively. Throughout this paper, differential system = distribution, [2, p. 10]. $\Delta_N^V (\subset \Delta_1^V)$ is the differential system of E defined by the orbits of N as a transformation group on E. g, g_h, g_N are the Lie algebras of G, H, N respectively. $\gamma = \{\Gamma\}$, $\gamma_1 = \{\Gamma_1\}$ are the sets of connections of ξ, ξ_1 respectively. $\varphi_\Gamma^0(\alpha)$ is the restricted holonomy group of Γ with reference point α.

In this paper we shall establish some properties of holonomy groups of connections of γ and γ_1. More precisely:

In γ the following equivalence relation is introduced ([4], [5]): Two connections $\Gamma, \Gamma' \in \gamma$ defined by the horizontal differential systems $\Delta_\Gamma^H, \Delta_{\Gamma'}^H$ are equivalent if $p_1'(\Delta_\Gamma^H) = p_1'(\Delta_{\Gamma'}^H)$. The equivalence class defined by $\Gamma \in \gamma$ has been denoted by Γ^*, [4, p. 381]. In the first section we shall make some remarks

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on the relations which exist between the holonomy groups of two equivalent connections of ξ.

If Δ is a differential system of E satisfying the relations

$$(1) \qquad \Delta \cap \Delta_1^V = 0 \,, \quad \Delta \oplus \Delta_1^V = \Delta^V \,, \quad \forall a \in H \,|\, R'_a(\Delta) = \Delta \,,$$

then we can define a map $h_\Delta : \gamma \to \gamma_1$, ([4], [5]). In the second section we shall establish a relation between the holonomy groups of two connections of γ, which have the same image in h_Δ.

In the third section we shall make some observations on the relations which exist between the holonomy groups of connections $\Gamma \in \gamma$ and $h_\Delta(\Gamma) \in \gamma_1$, Δ being fixed.

1. The holonomy groups of connections belonging to the same class Γ^*

Let $\mathscr{F}\mathscr{V}(E, \Delta_\Gamma^H), \mathscr{F}\mathscr{V}(E, \Delta_N^V)$ be the vector bundles defined over E as the base space, by the differential systems $\Delta_\Gamma^H, \Delta_N^V$ respectively. For the set Γ^* defined by Γ we have the theorem [5, Theorem 2]:

The set Γ^* is bijective with the set of homomorphism $\{T \,|\, T : \mathscr{F}\mathscr{V}(E, \Delta_\Gamma^H) \to \mathscr{F}\mathscr{V}(E, \Delta_N^V)\}$ satisfying the condition

$$(2) \qquad \forall a \in H \,|\, R'_a \circ T = T \circ R'_a \,.$$

Here we shall prove two theorems concerning the holonomy groups of connections of Γ^*. To this purpose we shall first establish

Lemma 1. *Suppose that $\xi = (E, B, p, F, G ; \mathscr{A})$ is a Steenrod bundle, and Δ is a differential system of E satisfying the following three conditions*

1) $\Delta \cap \Delta^V = 0$, 2) $\Delta \oplus \Delta^V = T(E)$,

3) *for every curve of B there is a horizontal lift with respect to p (horizontal = tangent to Δ), uniquely determined by an initial point (we shall say that the differential system Δ has the unique path lifting property with respect to p).*

Then Δ is involutive if and only if the horizontal lift with respect to p of an arbitrary closed zero homotopic curve of B is a closed curve of E.

Proof. Let Δ be an involutive differential system. Then for every point $x \in B$, there is an open neighborhood $U_x \subset B$ such that the intersection of the integral manifold of Δ, defined by an arbitrary point $\alpha \in p^{-1}(x)$, with $p^{-1}(U_x)$ be a differentiable manifold, diffeomorphic to U_x by p.

Let $x = x(t)$ be an arbitrary curve of B, closed in x_0 and zero homotopic. By applying the factorization lemma to $x = x(t)$ ([3, p. 47] or ([2, p. 284]) we see that every utilized lasso is contained in an above specified neighborhood U_x. The horizontal lifts of these lassos with respect to p and with a fixed point $\alpha_0 \in p^{-1}(x_0)$ as initial point, are also lassos. The product of these lassos is a closed curve, the horizontal lift of $x = x(t)$ with respect to p.

Conversely, we shall prove that if the horizontal lift with respect to p of an arbitrary closed zero homotopic curve of B is a closed curve in E, then Δ is an involutive differential system. Consider the vector fields $\partial/\partial x^i, i = 1, \cdots, n$, locally defined on B, and denote by $Z_i, i = 1, \cdots, n$, the horizontal lifts of these fields with respect to p. The vector fields $Z_i, i = 1, \cdots, n$, define a local base for Δ. Under our hypothesis we shall establish that $[Z_i, Z_j] = 0$. In order to prove this, consider the vector fields $\partial/\partial x^i, \partial/\partial x^j, -\partial/\partial x^i, -\partial/\partial x^j$ of B and their integral curves c_1, c_2, c_3, c_4. Using the geometrical interpretation of $[\partial/\partial x^i, \partial/\partial x^j]$, [1, p. 28], we see that the equality $[\partial/\partial x^i, \partial/\partial x^j] = 0$ implies that the curve defined by c_1, c_2, c_3, c_4 is closed and also zero homotopic. The horizontal lifs of c_1, c_2, c_3, c_4 with respect to p are denoted $\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4$, and these curves are tangent to $Z_i, Z_j, -Z_i, -Z_j$. The hypothesis of the lemma implies that the curve defined by these four curves $\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4$ are closed and hence $[Z_i, Z_j] = 0$.

Theorem 1. *Let $(\xi; \xi_0, \xi_1)$ be the refinement of ξ defined by H, and let Γ be a connection of ξ defined by the horizontal differential system Δ_Γ^H. If $\Delta_\Gamma = p_1'(\Delta_\Gamma^H)$, then Δ_Γ is a differential system of E/H and has the unique path lifting property with respect to p_0. Δ_Γ is involutive if and only if $\forall \alpha \in E \mid \varphi_\Gamma^0(\alpha) \subset H$.*

Proof. The relation $p = p_0 \circ p_1$ implies that Δ_Γ is a differential system of E/H.

Let $x = x(t)$ be an arbitrary curve of B, and α_1^0 a point in E/H so that $p_0(\alpha_1^0) = x(0)$. If $\alpha^0 \in E$ and $p_1(\alpha^0) = \alpha_1^0$, then there is a unique horizontal lift $\alpha = \alpha(t)$ of $x = x(t)$ with respect to p (horizontal $=$ tangent to Δ_Γ^H), so that $\alpha(0) = \alpha^0$. The curve $p_1(\alpha(t)) = \alpha_1(t)$ is a horizontal lift of $x = x(t)$ with respect to p_0, and $\alpha_1(0) = \alpha_1^0$ (horizontal $=$ tangent to Δ_Γ). If $\beta_1 = \beta_1(t)$ is another horizontal lift of $x = x(t)$ with respect to p_0 so that $\beta_1(0) = \alpha_1^0$, then $\alpha_1(t) = \beta_1(t)$. Indeed, let Δ be a differential system of E satisfying conditions (1). Then Δ and Δ_Γ^H determine a horizontal differential system of a connection Γ_1 in ξ_1. If $\beta_1 = \beta_1(t)$ is a horizontal lift of $x = x(t)$ with respect to p_0 (horizontal $=$ tangent to Δ_Γ), then the horizontal lift of $\beta_1 = \beta_1(t)$ with respect to p_1 (horizontal $=$ tangent to horizontal differential system of Γ_1) is a curve $\beta = \beta(t)$, $\beta(0) = \alpha^0$. This implies that $\beta = \beta(t)$ is a horizontal lift of $x = x(t)$ with respect to p (horizontal $=$ tangent to Δ_Γ^H). But from the unicity of the horizontal lift follow $\beta(t) = \alpha(t)$ and $p_1(\alpha(t)) = p_1(\beta(t))$ and hence $\alpha_1(t) = \beta_1(t)$. Thus we have established that Δ_Γ has the unique path lifting property.

Let $x = x(t)$ be a closed zero homotopic curve of B. Denote by $\alpha = \alpha(t)$ the horizontal lift of $x = x(t)$ with respect to p (horizontal $=$ tangent to Δ_Γ^H) so that $\alpha(0) = \alpha^0$, and by $\alpha_1 = \alpha_1(t)$ the horizontal lift $x = x(t)$ with respect to p_0 (horizontal $=$ tangent to Δ_Γ) so that $\alpha_1(0) = p_1(\alpha^0) = \alpha_1^0$. From Lemma 1, Δ_Γ is involutive if and only if $\alpha_1(1) = \alpha_1^0$. This condition is equivalent to the condition $\alpha(1) \in p_1^{-1}(\alpha_1^0)$. Since $\alpha(1) = \alpha^0 \cdot a$, a being an element of $\varphi_\Gamma^0(\alpha^0)$, the last condition is equivalent to the condition $\forall \alpha^0 \in E \mid \varphi_\Gamma^0(\alpha^0) \subset$ H.

Remark 1. The relation $\forall \alpha \in E \mid \varphi_\Gamma^0(\alpha) \subset H$ is equivalent to the relation $\forall \alpha \in E \mid \varphi_\Gamma^0(\alpha) \subset N$. For the former statement to be true it is sufficient that at least one point $\alpha \in E$ should exist such that $\varphi^0_\Gamma(\alpha) \subset N$. It means that the last statement of Theorem 1 may be substituted by the following one:

The differential system Δ_Γ of E/H is involutive if and only if there is at least one point $\alpha \in E$ such that $\varphi^0_\Gamma(\alpha) \subset N$.

Theorem 2. *Let* $(\xi\,;\,\xi_0, \xi_1)$ *be a refinement, and* Γ^* *a class of connections of* ξ. *If* $\Gamma, \Gamma' \in \Gamma^*$, *and* $\varphi^0_\Gamma(\alpha^0)$, $\varphi^0_{\Gamma'}(\alpha^0)$ *are the restricted holonomy groups of* Γ, Γ' *respectively with reference point* α^0, *then*

$$\varphi^0_\Gamma(\alpha^0)\cdot N = \varphi^0_{\Gamma'}(\alpha^0)\cdot N\ . \tag{3}$$

Proof. We shall establish the inclusion $\varphi^0_{\Gamma'}(\alpha^0) \subset \varphi^0_\Gamma(\alpha^0)\cdot N$. If $a' \in \varphi^0_{\Gamma'}(\alpha^0)$, then there is a zero homotopic curve $x = x(t) \subset B$, such that its horizontal lift with respect to p (horizontal = tangent to $\Delta^H_{\Gamma'}$) determined by $\alpha^0 \in E$ be a curve $\alpha = \alpha(t)$ such that $\alpha(0) = \alpha^0$, $\alpha(1) = \alpha^0\cdot a'$. In the same way the horizontal lift of $x = x(t)$ with respect to p (horizontal = tangent to Δ^H_Γ) determined by $\alpha^0 \in E$ is a curve $\alpha = \alpha(t)$ such that $\alpha(0) = \alpha^0$, $\alpha(1) = \alpha^0\cdot a$, $a \in \varphi^0_\Gamma(\alpha^0)$. But observing that $\Gamma, \Gamma' \in \Gamma^*$ we obtain $p_1(\alpha(t)) = p_1(\alpha'(t))$; the horizontal lift of $x = x(t)$ with respect to p_0, horizontal meaning that this lift is tangent to $\Delta_\Gamma = p_1'(\Delta^H_\Gamma) = p_1'(\Delta^H_{\Gamma'})$, is uniquely determined by $p_1(\alpha^0)$. This implies that for every $0 \le t \le 1$ we have $\alpha'(t) = \alpha(t)\cdot b(t)$, where $b(t) \in H$. This relation is satisfied for all the points $\alpha^0 \in p^{-1}(x(0))$. It follows that we have $a' = a\cdot b(1)$ and $\forall c \in G\,|\,\alpha'(t)\cdot c = \alpha(t)\cdot b(t)\cdot c$ so that $c^{-1}\cdot b(1)\cdot c \in H$. But this means that $b(1) \in N$ and that $\varphi^0_{\Gamma'}(\alpha^0) \subset \varphi^0_\Gamma(\alpha^0)\cdot N$.

The inclusion established implies also that $\varphi^0_{\Gamma'}(\alpha^0)\cdot N \subset \varphi^0_\Gamma(\alpha^0)\cdot N$. In the same way we can prove the inclusion $\varphi^0_\Gamma(\alpha^0)\cdot N \subset \varphi^0_{\Gamma'}(\alpha^0)\cdot N$ and hence $\varphi^0_\Gamma(\alpha^0)\cdot N = \varphi^0_{\Gamma'}(\alpha^0)\cdot N$.

Corollaries .1. *If among the connections of* Γ^* *there is one whose restricted holonomy group with reference point a fixed point in E is included in N, then the restricted holonomy groups of all connections of* Γ^*, *with arbitrary reference points, are included in N (in this case the differential system* $\Delta_\Gamma = p_1'(\Delta^H_\Gamma)$ *is involutive).*

2. *If the restricted holonomy groups of two connections of* Γ^* *with the same reference point in E contain the group N, then these two groups coincide.*

2. The holonomy groups of connections of a class $h_\Delta^{-1}(\Gamma_1)$

Let Δ be a differential system of E, which satisfies conditions (1). By means of Δ we can define a map $h_\Delta : \gamma \to \gamma_1$ [4, p. 380] as follows: If $\Gamma \in \gamma$, and the horizontal differential system of Γ is Δ^H_Γ, then $h_\Delta(\Gamma)$ is the element of γ_1, whose horizontal differential system is $\Delta \oplus \Delta^H_\Gamma$.

Denote by $\bar{\Delta}$ the differential system of E which is maximal among all the differential systems of E satisfying the conditions

$$\bar{\Delta} \subset \Delta\ , \qquad \forall a \in G\,|\,R_a'(\bar{\Delta}) = \bar{\Delta}\ , \tag{4}$$

and recall the following theorem [5, Theorem 7].

Let $\Gamma \in \gamma$ be a connection, and $\Gamma_1 = h_\Delta(\Gamma) \in \gamma_1$ its image in h_Δ. Then there is a bijection between the set $h_\Delta^{-1}(\Gamma_1)$ and the set of homomorphisms $\{T \mid T : \mathscr{F}\mathscr{V}(E, \Delta_\Gamma^H) \to \mathscr{F}\mathscr{V}(E, \bar{\Delta})\}$ satisfying the conditions

$$\forall a \in G \mid R'_a \circ T = T \circ R'_a \ . \tag{5}$$

($\mathscr{F}\mathscr{V}(E, \bar{\Delta})$ is the vector bundle defined over E, as the base space, by the differential system $\bar{\Delta}$.)

In what follows we shall utilize an invariant subgroup $N^\Delta \subset G$, defined by Δ in the following way:

Let M^Δ be the set of elements of Lie algebra g defined by the relation

$$M^\Delta = \bigcup_{\alpha \in E} (d\sigma_\alpha^{-1}(\bar{\Delta}(\alpha))) \ , \tag{6}$$

where σ_α is the map $\sigma_\alpha : b \in G \to \alpha \cdot b \in p^{-1}(p(\alpha))$, and $\bar{\Delta}$ is the differential system of E defined by (4). Utilizing the relation

$$\forall \alpha \in E \ , \qquad a \in G \mid R_{a^{-1}} \circ \sigma_{\alpha a} = \sigma_\alpha \circ \operatorname{adj} a \ , \tag{7}$$

we can prove that

$$M^\Delta \cap g_H = 0 \ , \qquad \forall a \in G \mid \operatorname{adj} a \cdot (M^\Delta) = M^\Delta \ . \tag{8}$$

Denote by g^Δ the smallest linear space of g, which includes the set M^Δ. From (8) follows $\forall a \in G \mid \operatorname{adj}(g^\Delta) = g^\Delta$. Utilizing the relation $[X, Y] = \lim_{t \to 0} t^{-1}(\operatorname{adj} a_t^{-1} \cdot X - X)$, [2, p. 41], where $X \in g^\Delta$ and $Y \in g$, we obtain that g^Δ is an ideal of g. Denote by N^Δ the invariant subgroup of G, which is connected and defined by g^Δ.

By means of N^Δ we shall have

Theorem 3. *Let Δ be a differential system of E satisfying conditions* (1), *and $h_\Delta : \gamma \to \gamma_1$ the map defined by Δ. For the restricted holonomy groups of two connections $\Gamma, \Gamma' \in h_\Delta^{-1}(\Gamma_1)$ with reference to the same point in E, we have the next relation*

$$\forall \alpha^0 \in E \mid \varphi_\Gamma^0(\alpha^0) \cdot N^\Delta = \varphi_{\Gamma'}^0(\alpha^0) \cdot N^\Delta \ . \tag{9}$$

Proof. In order to establish relation (9) it is sufficient to prove that $\varphi_\Gamma^0(\alpha^0) \subset \varphi_{\Gamma'}^0(\alpha^0) \cdot N^\Delta$ and that $\varphi_{\Gamma'}^0(\alpha^0) \subset \varphi_\Gamma^0(\alpha^0) \cdot N^\Delta$. If these two conditions are satisfied, then $\varphi_\Gamma^0(\alpha^0) \cdot N^\Delta \subset \varphi_{\Gamma'}^0(\alpha^0) \cdot N^\Delta$ and $\varphi_{\Gamma'}^0(\alpha^0) \cdot N^\Delta \subset \varphi_\Gamma^0(\alpha^0) \cdot N^\Delta$, and hence $\varphi_{\Gamma'}^0(\alpha^0) \cdot N^\Delta = \varphi_\Gamma^0(\alpha^0) \cdot N^\Delta$.

We shall prove that $\varphi_\Gamma^0(\alpha^0) \subset \varphi_{\Gamma'}^0(\alpha^0) \cdot N^\Delta$. Let a be an arbitrary element of $\varphi_\Gamma^0(\alpha^0)$. Then there is a curve $x = x(t)$ of B, which is closed in $p(\alpha^0)$ and zero homotopic, the horizontal lift $\alpha = \alpha(t)$ of which with respect to p (horizontal = tangent to Δ_Γ^H) satisfies the conditions $\alpha(0) = \alpha^0$, $\alpha(1) = \alpha^0 \cdot a$. Let us also consider the horizontal lift $\alpha' = \alpha'(t)$ of $x = x(t)$ with respect to p (horizontal = tangent to $\Delta^H_{\Gamma'}$). Evidently we have $\alpha(t)\cdot a(t) = \alpha'(t)$ where $a(t) \in G$. By means of Leibnitz formula we obtain

$$(10) \qquad \frac{d\alpha'}{dt} = \frac{d\alpha}{dt}\cdot a(t) + d\sigma_{\alpha(t)\cdot a(t)} \circ L'_{(a(t))^{-1}} \frac{da}{dt} .$$

Denote $d\alpha'/dt - (d\alpha/dt)\cdot a(t) = X_{\alpha(t)a(t)}$. Obviously, $X_{\alpha(t)a(t)} \in \bar{\Delta}(\alpha(t)a(t))$. If we denote $X_{\alpha(t)} = R'_{a^{-1}(t)}(X_{\alpha(t)a(t)})$, then we have $X_{\alpha(t)} \in \bar{\Delta}(\alpha(t))$. Relation (10) is equivalent to

$$(11) \qquad d\sigma_\alpha^{-1}(X_{\alpha(t)}) = R'_{(a(t))^{-1}} da/dt .$$

By the lemma [2, p. 69] and $d\sigma^{-1}_{\alpha(t)}(X_{\alpha(t)}) \in M^\Delta$, $\forall t \in [0, 1]$, we obtain a unique curve $a(t) \in N^\Delta$ such that $a(0) = e$, satisfying (11), differentiable of at least order 1. For this curve relation (10) is satisfied and hence $\alpha'(t) = \alpha(t)\cdot a(t)$. It follows that $\alpha(1)\cdot a(1) = \alpha'(1)$ and consequently $a\cdot a(1) = a'$ where a' is the element of $\varphi^0_{\Gamma'}(\alpha^0)$ defined by the horizontal lift of $x = x(t)$ with respect to p (horizontal = tangent to $\Delta^H_{\Gamma'}$). In conclusion we have proved that to each element $a \in \varphi^0_{\Gamma'}(\alpha^0)$ we can associate an element $a_1 \in N^\Delta$ such that $aa_1 \in \varphi^0_{\Gamma'}(\alpha^0)$. Then $a = (aa_1)\cdot a_1^{-1} \in \varphi^0_{\Gamma'}(\alpha^0)\cdot N^\Delta$ and hence $\varphi^0_\Gamma(\alpha^0) \subset \varphi^0_{\Gamma'}(\alpha^0)\cdot N^\Delta$. In the same way we can prove the inclusion $\varphi^0_{\Gamma'}(\alpha^0) \subset \varphi^0_\Gamma(\alpha^0)\cdot N^\Delta$. According to the first part of this proof, we obtain relation (9).

Remark 2. Let M be the union of sets M^Δ for all Δ satisfying conditions (1). If g_0 is the smallest linear space of g which contains M, then g_0 is an ideal of g, which defines an invariant subgroup N_0 of g. Evidently for every Δ we have $N^\Delta \subset N_0$, and if $\Gamma' \in h_\Delta^{-1}(h_\Delta(\Gamma))$ then

$$(12) \qquad \forall \alpha \in E \,|\, \varphi^0_\Gamma(\alpha^0)\cdot N_0 = \varphi^0_{\Gamma'}(\alpha^0)\cdot N_0 .$$

If N_1 is an invariant subgroup of G such that $N_1 \cap H = \{e\}$, then $g_{N_1} \subset M$ and $N_1 \subset N_0$, where g_{N_1} is the Lie algebra of N_1.

Remark 3. Relation (6) shows that the subgroup N^Δ is defined by the differential system $\bar{\Delta}$ satisfying conditions (4). Accordingly we can obtain the following result: Let $\Gamma, \Gamma' \in \gamma$ be a pair of connections. If we consider the homomorphism $T: \mathscr{F}\mathscr{V}(E, \Delta^H_\Gamma) \to \mathscr{F}\mathscr{V}(E, \Delta^V)$ defined by the couple Γ, Γ' [5, Theorem 1] and if $\operatorname{Im} T \cap \Delta^V_1 = 0$, then between the restricted holonomy groups of Γ, Γ' we have relation (9) where N^Δ is substituted by the invariant subgroup $N^{J_m T}$ of G.

3. The holonomy groups of connections $\Gamma \in \gamma$ and $h_\Delta(\Gamma) \in \gamma_1$

Let Δ be a differential system of E satisfying conditions (1). If $\Gamma \in \gamma$ and

$\Gamma_1 = h_\Delta(\Gamma)$, then for the holonomy groups of the two connections we have [4, p. 380]

$$(13) \qquad \forall \alpha \in E \,|\, \varphi_\Gamma(\alpha) \cap H \subset \varphi_{\Gamma_1}(\alpha) \subset H \ .$$

Lemma 2. *Let* $(\xi; \xi_0, \xi_1)$ *be the refinement of* ξ *defined by* H. *For an arbitrary point* $x \in B$, *the restriction of* ξ_1 *to* $p_0^{-1}(x)$ *is a principal differentiable bundle* $\xi_x = (p^{-1}(x), p_0^{-1}(x), p_1/p^{-1}(x), H; \mathscr{A}_x)$ *isomorphic to* $(G, G/H)$, [6, p. 39].

Proof. ξ_x is a principal differentiable bundle with $p^{-1}(x)$ as the total space, $p_0^{-1}(x)$ as the base space, $p_1/p^{-1}(x)$ as the projection, H as the type fiber and structural group. The trivialization atlas $\mathscr{A}_x$ is defined by means of the trivialization atlases of ξ and $(G, G/H)$ in the following way: If (U, φ_U) is a trivialization map of ξ so that $x \in U$, and if (u, f) is a trivialization map of $(G, G/H)$, then define a map $(V, \psi) \in \mathscr{A}_x$ by the relations: $V = U_1 \cap p_0^{-1}(x)$ where $U_1 = \varphi_{U;1}(U \times u)$ and $\psi = \varphi_{u;12}(V \times H)$ (see [4, Relation (19), p. 372]).

We can observe that for an arbitrary $a \in H$ the right translation of ξ_x coincides with the right translation of ξ_1,

Let α be a fixed element of $p^{-1}(x)$. Define an isomorphism $(i_\alpha, i'_\alpha): \xi_x \to (G, G/H)$ by the relations

$$i_\alpha : \beta = \alpha \cdot a \to a \in G \ ; \qquad i'_\alpha = 1_H \ .$$

Then we have $\forall b \in G \,|\, i_\alpha(\beta \cdot b) = i_\alpha(\beta) \cdot i'_\alpha(b)$.

Remark 4. The differential system Δ of E satisfying conditions (1) defines a connection Γ^x_Δ in every ξ_x. The holonomy group of Γ^x_Δ with reference point $\alpha \in p^{-1}(x)$ is denoted by $\varphi_{(x,\Delta)}(\alpha)$.

Remark 5. If the differential system Δ of E satisfying conditions (1) is fixed and if Γ_1 is a connection at ξ_1 such that $\Delta^H_{\Gamma_1} \supset \Delta$, then

$$(14) \qquad \forall \alpha \in E \,|\, \varphi_{(x,\Delta)}(\alpha) \subset \varphi_{\Gamma_1}(\alpha) \subset H \ , \qquad p(\alpha) = x \ .$$

Remark 6. 1. Relation (13) shows that if for a point $\alpha \in E$ the holonomy group $\varphi_\Gamma(\alpha)$ of a connection $\Gamma \in \gamma$ satisfies the conditions $\varphi_\Gamma(\alpha) \supset H$, then $h_\Delta(\Gamma)$ for all Δ has H as a holonomy group.

2. Relation (14) shows that if Δ is a differential system of E satisfying conditions (1) such that the connection defined by Δ in ξ_x (x arbitrary in B) has H as holonomy groups, then all connections Γ_1 satisfying the condition $\Delta \subset \Delta^H_{\Gamma_1}$ have H as holonomy group.

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